[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1264 **D**

Unique Paper Code : 2354001001

Name of the Paper : GE: Fundamentals of Calculus

Name of the Course : **Common Prog. Group**

Semester : I

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory and carry equal marks.
3. This question paper has six questions.
4. Attempt any **two** parts from each question.

1. (a) (i) Establish that $\lim\limits_{x\to 0} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ does not exist.

(ii) Examine the continuity of the function

$$g(x) = \begin{cases} -x^2 & , \text{ if } x \le 0 \\ 5x - 4 & , \text{ if } 0 < x \le 1 \\ 4x^2 - 3x, & \text{ if } 1 < x < 2 \\ 3x + 4 & , \text{ if } x \ge 2 \end{cases}$$

at $x = 0, 1, 2$ and discuss their type of discontinuities, if any.

(b) Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Also prove that if $x^y = e^{x-y}$, then $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

(c) Find the n^{th} derivatives of $f(x) = e^{ax}\cos^2 bx$ and $g(x) = \sin 5x \sin 3x$.

2. (a) If $y = e^{m\sin^{-1}x}$, then show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Also find $y_n(0)$.

(b) Let $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ and $v = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$.

Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0 \text{ and } x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \tan V$$

(c) If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, then prove that $\frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} + \frac{\partial^2 V}{\partial^2 z} = m(m+1)r^{m-2}$.

3. (a) State and prove Rolle's theorem. Verify it for the function

$f(x) = x^3 - 6x^2 + 11x - 6$ in the domain [1,3].

(b) State Lagrange's mean value theorem. Use it to show that

$$\frac{x}{1+x} < \log(1+x) < x, \text{ for all } x > 0.$$

(c) Verify Cauchy's mean value theorem for the following pair of functions:

(i) $f(x)=\frac{1}{x^2}$ and $g(x)=\frac{1}{x}$ in the domain $[2,5]$.

(ii) $f(x) = \sin x$ and $g(x) = \cos x$ in the domain $[0, \pi/2]$.

(iii) $f(x) = e^x$ and $g(x) = e^{-x}$ in the domain $[1,4]$.

4. (a) Find the range of x for which the series $a + ax + ax^2 + \ldots + ax^{n-1} + \ldots$ is convergent, where a is a nonzero real number. Verify whether the series $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\cdots$ is convergent or not.

(b) Find the Taylor's series for $f(x) = \sin x$ and $g(x) = \cos x$.

(c) Evaluate the following :

(i) $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$.

(ii) $\lim_{x\to 0}\left(\dfrac{\tan^2 x - x^2}{x^2 \tan^2 x}\right)$.

5. (a) Determine the intervals of concavity and points of inflection of the curve $y = 3x^5 - 40x^3 + 3x - 20$. Also use both first and second derivative tests to show that $f(x) = x^3 - 3x + 3$ has relative minimum at $x = 1$.

(b) Find asymptotes of the curve:
$y^3 - 2xy^2 - x^2y + 2x^3 + 2x^2 - 3xy + x - 2y + 1 = 0$.

(c) Determine the intervals of concavity and points of inflection of the curve $y = e^{-x^2}$. Also, show that the points of inflection of the curve $y = -(x-3)\sqrt{(x-5)}$ lies on the line $3x = 17$.

6. (a) Sketch a graph of $y = \dfrac{x}{x^2+4}$ and identify the locations of all asymptotes, intercepts, relative extrema and inflection points.

(b) Locate the critical points and identify which critical points are stationary points for the functions:

(i) $f(x) = 4x^4 - 16x^2 + 17$

(ii) $g(x) = 3x^4 + 12x$

(iii) $h(x) = 3x^{5/3} - 15x^{2/3}$.

(c) Trace the curve $r = 2(1 + \cos\theta)$.