[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 2811 **A**

Unique Paper Code : 62351201

Name of the Paper : Algebra

Name of the Course : **B.A. (Prog.)**

Semester : II

Duration : 3 hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions carry equal marks.

1. (a) Show that the set $W = \{(a_1, a_2, a_3): a_1 - 2a_2 + a_3 = 0;\ a_1, a_2, a_3 \in R\}$ is a subspace of the vector space $R^3(R)$.

(b) Let $\{a, b, c\}$ be a basis of $R^3(R)$. Show that the set $\{a + b, b + c,\ c + a\}$ is also a basis of $R^3(R)$

(c) Define linearly independent set of vectors. Show that the following set of vectors $\{(1,2,3),(3,-1,0)\}$ in $R^3(R)$ is linearly independent.

(d) Let $S = \{(a,0,0): a \in R\}$ and $T = \{(0,b,0): b \in R\}$ be subsets of R^3. Show that S and T are subspaces of R^3.

2. (a) Reduce the following matrix to triangular form by elementary row operations and hence find the rank:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 5 & -1 & 6 \end{bmatrix}.$$

(b) Solve the following system of equations:

$$x + 2y + 3z = 0$$
$$2x + 4y + 7z = 0$$
$$3x + 6y + 10z = 0$$

(c) Find the eigen values of the following matrix:

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

(d) Verify that the following matrix satisfies its characteristic equation:

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}.$$

3. (a) If n is a positive integer, show that $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{4}$.

(b) Prove that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

(c) Solve the equation

$$z^5 + 1 = 0.$$

(d) Find the sum

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta.$$

4. (a) Solve the equation $x^3 - 5x^2 - 16x + 80 = 0$, the sum of two of the roots being zero.

(b) If α, β, γ be the roots of the equation $x^3 - 7x + 7 = 0$, find the value of $\sum \alpha^4$.

(c) Find the sum of the cubes of the roots of the equation $x^3 + 5x^2 - 6x + 3 = 0$.

(d) Solve the equation $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being in the ratio 3: 4.

5. (a) Show that $H = \left\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \neq 0, a, b \in \mathbb{R}\right\}$ is a subgroup of the multiplicative group of 2×2 non-singular matrices over $\mathbb{R}$.

(b) Show that the set $\mathbb{Q}$ of all rational numbers other than 1 is an Abelian group with respect to the binary composition $a * b = a + b - ab$.

(c) State and prove Lagrange's theorem.

(d) Express the following permutation as a product of transpositions and hence determine whether it is odd or even:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

6. (a) Prove that intersection of two subrings of a ring $(R, +, \cdot)$ is a subring of R.

(b) Show that in a group G, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for all $a, b \in G$.

(c) Compute $4^{122} \pmod{11}$ and $2^{2022} \pmod{21}$ using Euler's theorem.

(d) Let R be a ring. The *center* of R is the set $\{x \in R : ax = xa \ \forall\, a \in R\}$. Prove that the center of a ring is a subring.