[This question paper contains 6 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1395 A

Unique Paper Code : 32351403

Name of the Paper : BMATH-410 – Ring Theory and Linear Algebra – I

Name of the Course : CBCS (LOCF) B.Sc. (H) Mathematics

Semester : IV

Duration : 3.30 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Define zero divisors in a ring. Let R be the set of all real valued functions defined for all real numbers under function addition and multiplication. Determine all zero divisors of R. (6½)

P.T.O.

(b) What is nilpotent element? If a and b are nilpotent elements of a commutative ring, show that a + b is also nilpotent. Give an example to show that this may fail if the ring R is not commutative. (6½)

(c) Let R be a commutative ring with unity. Prove that U(R), the set of all units of R, form a group under multiplication of R. (6½)

(d) Determine all subrings of $\mathbb{Z}$, the set of integers. (6½)

2. (a) Define centre of a ring. Prove that centre of a ring R is a subring of R. (6)

(b) Suppose R is a ring with $a^2 = a$, for all $a \in R$. Show that R is a commutative ring. (6)

(c) Show that any finite field has order p^n, where p is prime. (6)

(d) Let R be a ring with unity 1. Prove that if 1 has infinite order under addition, then CharR = 0, and if 1 has order n under addition, then CharR = n. (6)

3. (a) Let R be a commutative ring with unity and let A be an ideal of R. Then show that R/A is a field if and only if A is maximal ideal. (6½)

(b) Prove that $I = \langle 2 + 2i \rangle$ is not prime ideal of $\mathbb{Z}[i]$.

How many elements are in $\frac{\mathbb{Z}[i]}{I}$? What is the characteristic of $\frac{\mathbb{Z}[i]}{I}$? (6½)

(c) In $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, let $I = \{f(x) \in \mathbb{Z}[x] \mid f(0) = 0\}$. Prove that I is not a maximal ideal. (6½)

(d) Let $\mathbb{R}[x]$ denote the ring of polynomials with real coefficients and let $\langle x^2 + 1 \rangle$ denote the principal ideal generated by $x^2 + 1$. Then show that

$$\frac{\mathbb{R}[x]}{\langle x^2+1\rangle} = \{g(x) + \langle x^2+1\rangle \mid g(x) \in \mathbb{R}[x]\} =$$

$$\{ax + b + \langle x^2+1\rangle \mid a, b \in \mathbb{R}\}$$

(6½)

4. (a) If R is a ring with unity and the characteristic of R is $n > 0$, then show that R contains a subring isomorphic to $\mathbb{Z}_n$ and if the characteristic of R is 0 then R contains a subring isomorphic to $\mathbb{Z}$. (6)

(b) Determine all ring homomorphism from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$. (6)

(c) Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 11 if and only if $a_0 - a_1 + a_2 - \cdots + (-1)^k a_k$ is divisible by 11. (6)

(d) Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism. (6)

5. (a) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 + W_2$ is a smallest subspace of V that contains both W_1 and W_2. (6)

(b) For the following polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as linear combination of other two. (6)

$$\{x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3\}.$$

(c) Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k $(1 \leq k < n)$. (6)

(d) Let $W_1 = \{(a, b, 0) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}$ and $W_2 = \{(0, b, c)$ E $\mathbb{R}^3 : b, c \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Determine $\dim(W_1)$, $\dim(W_2)$, $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$. Hence deduce that $W_1 + W_2 = \mathbb{R}^3$. Is $\mathbb{R}^3 = W_1 \oplus W_2$? (6)

6. (a) Let V and W be finite-dimensional vector spaces having ordered bases β and γ respectively, and let $T : V \to W$ be linear. Then for each $u \in V$, show

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta. \quad (6½)$$

(b) Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be linear transformation defined by

$$T(a + bx + cx^2) = (a - c) + (a - c)x + (b - a)x^2 + (c - b)x^3.$$

Find null space N(T) and range space R(T). Also verify Rank-Nullity Theorem. (6½)

(c) For the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix}$ and ordered basis $\beta = \{(1, 1, 0), (0, 1, 1), (1, 2, 2)\}$, find $[L_A]_\beta$. Also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$. (6½)

(d) Let V and W be vector spaces and let T: V → W be linear. Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W. (6½)