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## [This question paper contains 4 printed pages.]

## Your Roll No.....

Sr. No. of Question Paper: 1132

Unique Paper Code : 32351201

Name of the Paper : BMATH203 - Real Analysis

Name of the Course : B.Sc. (H) Mathematics

Semester : II

Duration: 3 Hours Maximum Marks: 75

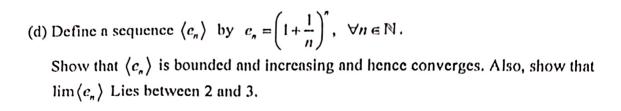
## Instructions for Candidates

- 1. Write your Roll No. on the top immediately on receipt of this question paper.
- 2. All questions are compulsory.
- 3. Attempt any two parts from each question.
- 4. All questions carry equal marks.
- 1. (a) Let S be a non-empty bounded set of  $\mathbb{R}$ . Let b < 0 and let  $bS = \{bs \mid s \in S\}$ . Prove that inf(bS) = b sup S and sup(bS) = b inf S.
  - (b) If y is a positive real number, show that there exists  $n_y \in \mathbb{N}$  such that

$$n_y - 1 \le y < n_y$$

(c) Let X be a non- empty set. Let f and g be defined on  $\mathbb R$  and have bounded ranges in  $\mathbb R$ . Show that

$$\sup \{ f(x) + g(x) \mid x \in X \} \le \sup \{ f(x) \mid x \in X \} + \sup \{ g(x) \mid x \in X \}.$$



- 2. (a) State and prove Density theorem.
  - (b) Let A and B be bounded non- empty subsets of  $\mathbb{R}$  and let  $A+B=\{a+b \mid a\in A,b\in B\}$ . Prove that  $\sup(A+B)=\sup A+\sup B$  and  $\inf(A+B)=\inf A+\inf B$
  - (c) Let  $I_n = \left[0, \frac{1}{n}\right]$ ,  $n \in \mathbb{N}$ . Show that  $\{I_n, n \in \mathbb{N}\}$  is a nested sequence of intervals and  $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$ .
  - (d) Examine the convergence of the series  $\sum_{n=1}^{\infty} ne^{-n^2}$
- 3. (a) State and prove Monotone Convergence Theorem.
  - (b) Let  $(x_n)$  be a sequence of positive real numbers such that  $\lim_{n\to\infty} (x_n^{1/n}) = L$  exists. Prove that if L < 1, then  $(x_n)$  converges and  $\lim_{n\to\infty} (x_n) = 0$ .
  - (c) Prove that  $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$ .
  - (d) Use the definition of the limit to show that  $\lim_{n\to\infty} (x_n) = 0$ , where  $x_n = 1/\ln(n+1)$ , for  $n \in \mathbb{N}$ . Also find  $K \in \mathbb{N}$  for  $\varepsilon = \frac{1}{10}$  such that  $|x_n 0| < \varepsilon$ ,  $\forall n \ge K$ .
- 4. (a) Let  $X = (x_n)$  and  $Y = (y_n)$  be sequences of real numbers that converge to x and y respectively and if  $y \neq 0$ . Then the quotient sequence X/Y converges to x/y.
  - (b) State and prove Squeeze Theorem. Also find  $\lim_{n\to\infty} \left(\frac{\sin n}{n}\right)$

- (c) State Cauchy Convergence Criterion for Sequences. Let  $X = (x_n)$  be defined by  $x_1 = 1$ ,  $x_2 = 2$  and  $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for n > 2. Prove that the sequence X is convergent.
- (d) Discuss the convergence of the sequence  $(x_n)$ , where  $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ , for each  $n \in \mathbb{N}$ .
- (a) Suppose the k th partial sum of  $\sum_{k=1}^{\infty} x_k$  is  $s_k = \frac{k}{k+1}$ . Find the corresponding series and general term  $x_{\bullet}$ . Prove that the series converges and then find the limit.
  - (b) Prove that the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges (despite the fact that  $\lim_{n \to \infty} \frac{1}{n} = 0$ ).
  - (c) Test for convergence, the following series:

(i) 
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$
 (ii)  $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \cdots$ 

- (d) Show that the series  $1 \frac{1}{2^p} + \frac{1}{3^p} \frac{1}{4^p} + \cdots, p > 0$  converges absolutely for p > 1 and conditionally for 0 .
- 6. (a) Prove that if  $\sum_{n=1}^{\infty} a_n$  is a series of positive terms and that its partial sums are bounded, then  $\sum_{n\geq 1} a_n$  converges. Show that this is not necessarily true if  $\sum_{n\geq 1} a_n$  is not a series of positive terms.
  - (b) State and prove the limit comparison test.
  - (c) Test for convergence, the following series:

(i) 
$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots + \frac{1}{n^n} + \dots$$
  
(ii)  $\sum_{n=1}^{\infty} 3^{-n-(-1)^n}$ 

(ii) 
$$\sum_{n=1}^{\infty} 3^{-n-(-1)^n}$$

(d) Define absolute and conditional convergence of an alternating series. Show that the series  $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$  is conditionally convergent but not absolutely.