9 MAY to 7 JUN

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[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1132 A

Unique Paper Code : 32351201

Name of the Paper : BMATH203 – Real Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.

1. (a) Let S be a non- empty bounded set of $\mathbb{R}$. Let $b<0$ and let $bS=\{bs \mid s\in S\}$. Prove that $inf(bS)=b\,sup\,S$ and $sup(bS)=b\,inf\,S$.

(b) If y is a positive real number, show that there exists $n_y \in \mathbb{N}$ such that

$$n_y - 1 \le y < n_y$$

(c) Let X be a non- empty set. Let f and g be defined on $\mathbb{R}$ and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x)+g(x) \mid x\in X\} \le \sup\{f(x) \mid x\in X\}+\sup\{g(x) \mid x\in X\}.$$

P.T.O.

(d) Define a sequence $\langle e_n \rangle$ by $e_n = \left(1+\frac{1}{n}\right)^n$, $\forall n \in \mathbb{N}$.

Show that $\langle e_n \rangle$ is bounded and increasing and hence converges. Also, show that $\lim \langle e_n \rangle$ Lies between 2 and 3.

2. (a) State and prove Density theorem.

(b) Let A and B be bounded non- empty subsets of $\mathbb{R}$ and let $A+B=\{a+b \mid a \in A, b \in B\}$. Prove that $\sup(A+B)=\sup A+\sup B$ and $\inf(A+B)=\inf A+\inf B$

(c) Let $I_n = \left[0, \frac{1}{n}\right]$, $n \in \mathbb{N}$. Show that $\{I_n, n \in \mathbb{N}\}$ is a nested sequence of intervals and $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$.

(d) Examine the convergence of the series $\sum_{n=1}^{\infty} ne^{-n^2}$

3. (a) State and prove Monotone Convergence Theorem.

(b) Let (x_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} (x_n^{1/n}) = L$ exists. Prove that if $L < 1$, then (x_n) converges and $\lim_{n\to\infty} (x_n) = 0$.

(c) Prove that $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$.

(d) Use the definition of the limit to show that $\lim_{n\to\infty} (x_n) = 0$, where $x_n = 1/\ln(n+1)$, for $n \in \mathbb{N}$. Also find $K \in \mathbb{N}$ for $\varepsilon = \frac{1}{10}$ such that $|x_n - 0| < \varepsilon, \forall n \geq K$.

4. (a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y respectively and if $y \neq 0$. Then the quotient sequence X/Y converges to x/y.

(b) State and prove Squeeze Theorem. Also find $\lim_{n\to\infty} (\frac{\sin n}{n})$

(c) State Cauchy Convergence Criterion for Sequences.
Let $X = (x_n)$ be defined by $x_1 = 1, x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n > 2$. Prove that the sequence X is convergent.

(d) Discuss the convergence of the sequence (x_n), where
$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{n^2}$, for each $n \in N$.

5. (a) Suppose the k th partial sum of $\sum_{n=1}^{\infty} x_n$ is $s_k = \frac{k}{k+1}$. Find the corresponding series and general term x_n. Prove that the series converges and then find the limit.

(b) Prove that the harmonic series $\sum \frac{1}{n}$ diverges (despite the fact that $\lim \frac{1}{n} = 0$).

(c) Test for convergence, the following series:

(i) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (ii) $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \cdots$

(d) Show that the series $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, p > 0$ converges absolutely for $p > 1$ and conditionally for $0 < p \leq 1$.

6. (a) Prove that if $\sum_{n\geq 1} a_n$ is a series of positive terms and that its partial sums are bounded, then $\sum_{n\geq 1} a_n$ converges. Show that this is not necessarily true if $\sum_{n\geq 1} a_n$ is not a series of positive terms.

(b) State and prove the limit comparison test.

(c) Test for convergence, the following series:

(i) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots$

(ii) $\sum_{n=1}^{\infty} 3^{-n-(-1)^n}$

(d) Define absolute and conditional convergence of an alternating series. Show that the series $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent but not absolutely.