[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1342 A

Unique Paper Code : 32351202

Name of the Paper : Differential Equations

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Use of non-programmable scientific calculator is allowed.

Q 1 Attempt any *three* parts. Each part is of 5 marks.

(a) Solve the differential equation

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$$

(b) Solve the initial value problem

$$x^2 \frac{dy}{dx} + xy = \frac{y^3}{x}, \qquad y(1) = 1.$$

(c) Determine the constant A such that the equation

$$\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right) dx + \left(\frac{1}{x^2} - \frac{1}{x}\right) dy = 0,$$

is exact and hence solve it.

(d) Solve the differential equation

$$(x^2 + xy)dy - (x^2 + y^2)dx = 0.$$

(c) Find the general solution of the differential equation

$$yy'' + (y')^2 = yy'.$$

Q 2 Attempt any *two* parts. Each part is of 6 marks.

(a) In a certain bacteria culture the rate of increase in the number of bacteria is proportional to the number present.

(i) If the number triples in 5 hrs, how many will be present after 10 hrs?

(ii) When will the number present be 10 times the number initially present?

(b) Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity?

(c) Suppose that $\rho = 0.075$ (in fps units with $g = 32\ ft/s^2$) in equation

$$\frac{dv}{dt} = -g + \rho v^2 = -g(1 - \frac{\rho}{g}v^2)$$

where v is the velocity, for a paratrooper falling with parachute open. If he jumps from an altitude of 10,000 ft and opens his parachute immediately, what will be his terminal speed? How long will it take him to reach the ground?

(d) Consider the American system of two lakes. Lake Erie feeding into Lake Ontario. Assuming that volume in each Lake to remain constant and that Lake Erie is the only source of pollution for Lake Ontario.

(i) Write down the differential equation describing the concentration of pollution in each of two lakes, using the variable V for volume F for flow C(t) for concentration at time t and subscripts 1 for lake Erie and 2 for Lake Ontario.

(ii) Suppose that only UN polluted water flows into lake Erie. How does this change the proposed model?

(iii) Solve the system of equations to get expression for the pollution concentration $C_1(t)$ and $C_2(t)$.

Q 3 Attempt any *two* parts. Each part is of 6 marks.

(a) Write down the word equations along with compartment diagrams that describe the density-dependent growth of a population with a constant harvesting rate. From the word equations, develop the differential equation by defining all variables and parameters as required.

(b) Let the dry weight of some plant at time t be denoted by $x(t)$ and suppose this plant feeds off a fixed amount of some single substrate or a nutrient medium, for which the amount remaining at a time t is denoted by $S(t)$. Assume that the growth rate of the plant is proportional to its dry weight as well as to the amount of nutrients available and that no material is lost in the conversion of S into x.

(i) Staring with the word equation, model the rate of plant growth dx/dt with an initial plant biomass of x_0 and with x_f the amount of plant material associated with $S = 0$.

(ii) By solving the differential equation, find the dry weight of the plant at any time t.

(c) Find the equilibrium solutions of the logistic differential equation and also discuss its stability.

(d) Consider the model

$$\frac{dx}{dt} = I - k_1 x, \qquad x(0) = 0$$

$$\frac{dy}{dt} = k_1 x - k_2 y, \quad y(0) = 0,$$

where $k_1, k_2 > 0$ and I is the amount of drug released into the GI tract in each time step. The levels of the drug in the GI tract and bloodstreams are $x(t)$ and $y(t)$ respectively.

Find the solution expressions for $x(t)$ and $y(t)$ which satisfy this pair of differential equations.

Q 4 Attempt any *two* parts. Each part is of 6 marks.

(a) Find the general solution of the differential equation

$$x^3 \frac{d^3y}{dx^3} - 3x^2 \frac{d^2y}{dx^2} + 6x \frac{dy}{dx} - 6y = 0.$$

(b) Show that $y = x^3$ is a solution of $y'' = 6x^4$, but that if $c^2 \neq 1$, then $y = Cx^3$ is not a solution.

(c) Use the Wronskian to prove that the functions

$$f(x) = e^x, \; g(x) = Cos\, x, \; k(x) = Sin\, x,$$

are linearly independent on the real line.

(d) Use method of undetermined coefficient to find particular solution of differential

$$y'' + 9y = 2Cos\, 3x + 3Sin\, 3x.$$

Q 5 Attempt any *two* parts. Each part is of 6 marks.

(a) Find the general solution of the differential equation

$$y^{(4)} + 18y'' + 81y = 0.$$

(b) Solve the initial value problem

$$y^{(3)} + 10y'' + 25y' = 0; \quad y(0) = 3, y'(0) = 4, y''(0) = 5.$$

(c) Find the general solution of the Euler's equation

$$x^3y''' + 6x^2y'' + 7xy' + y = 0.$$

(d) Use the method of variation of parameters to find the solution of the differential equation

$$y'' - 2y' - 8y = 3e^{-2x}.$$

Q 6 Attempt any *two* parts. Each part is of 6 marks.

(a) A simple mathematical model of the battle between two armies is given by the coupled differential equations

$$\frac{dR}{dt} = -a_1B - c, \qquad \frac{dB}{dt} = -a_2R,$$

where a_1, a_2 and c are positive constants.

(i) If the initial number of red soldiers is r_0 and the initial number of blue soldiers is b_0, use the chain rule to find a relationship between B and R.

(ii) For $a_1 = a_2 = c = 0.01$, give a sketch of typical phase-plane trajectories and deduce the direction of travel along the trajectories.

(b) Consider a disease, where all those who recovered from the disease are immune now. By making appropriate assumptions develop a model with a system of differential equations describing this epidemic model.

(c) The predator-prey equations with additional deaths by DDT are

$$\frac{dX}{dt} = \beta_1X - c_1XY - p_1X, \qquad \frac{dY}{dt} = -\alpha_2Y + c_2XY - p_2Y,$$

where all parameters are positive constants.

(i) Find all the equilibrium points.

(ii) What effect does the DDT have on the non-zero equilibrium populations compared with the case when there is no DDT?

(iii) Show that the predator fraction of the total average prey population is given by

$$f = \frac{1}{1 + \left(\frac{c_1(\alpha_2 + p_2)}{c_2(\beta_1 - p_1)}\right)}.$$

What happens to this proportion f as the DDT kill rates, p_1 and p_2, increase?

(d) Formulate a mathematical model for a predator-prey system where the prey protect their young from the predators. The model should have three dependent variables: $X_1(t)$, the juvenile prey numbers; $X_2(t)$, the adult prey numbers; and $Y(t)$, the predator numbers. In this model, it is assumed that the juvenile prey are completely sheltered from the predators.

(100)