[This question paper contains 8 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 4512 E

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt **two** parts from each question.

1. (a) Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \text{ and } 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$. (6)

(b) (i) Prove that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

as z tends to 0 does not exist.

(ii) Show that

$$\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4 .$$

(3+3=6)

(c) Show that the following functions are nowhere differentiable.

(i) $f(z) = z - \bar{z}$,

(ii) $f(z) = e^y \cos x + i e^y \sin x$. (3+3=6)

(d) (i) If a function f(z) is continuous and nonzero at a point z_0, then show that $f(z) \neq 0$ throughout some neighborhood of that point.

(ii) Show that the function $f(z) = (z^2 - 2)e^{-x}e^{-iy}$ is entire. (3+3=6)

2. (a) (i) Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp\ \exp\ (2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}.$$

(ii) Find the value of z such that

$$e^z = 1 + \sqrt{3}i$$ (3.5+3=6.5)

(b) Show that

(i) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;

(ii) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\pi i$ ($n = 0 \pm 1, \pm 2, \ldots$). (3.5+3=6.5)

(c) Show that

(i) $\log\log(i^2) = 2\log i$ where

$$\log z = \ln r + i\theta \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right).$$

(ii) $\log\log(i^2) \neq 2\log i$ where

$$\log z = \ln r + i\theta \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right).$$

(3.5+3=6.5)

(d) Find all zeros of sinz and cosz. (3.5+3=6.5)

3. (a) State Fundamental theorem of Calculus.

Evaluate the following integrals to test if Fundamental theorem of Calculus holds true or not :

(i) $\int_0^{\pi/2} \exp(t+it)\,dt$

(ii) $\int_0^1 (3t-i)^2\,dt$ (2+2+2=6)

(b) Let y(x) be a real valued function defined piecewise on the interval $0 \le x \le 1$ as

$$y(x) = x^3 \sin(\pi/x),\ 0 < x \le 1 \text{ and } y(0) = 0$$

Does this equation $z = x + iy,\ 0 \le x \le 1$ represent

(i) an arc

(ii) A smooth arc. Justify.

Find the points of intersection of this arc with real axis. (2+2+2=6)

(c) For an arbitrary smooth curve C: $z = z(t),\ a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integral depends only on the end points of C.

State if it is independent of the arc under consideration or not?

Also, find its value around any closed contour. (3+1+2=6)

(d) Without evaluation of the integral, prove that

$\left|\int_C \frac{1}{z^2+1} dz\right| \le \frac{1}{2\sqrt{5}}$ where C is the straight line segment from 2 to 2 +i. Also, state the theorem used. (4+2=6)

4. (a) Use the method of antiderivative to show that

$\int_C (z-z_0)^{n-1} dz = 0$, n = ±1, ±2,-- where C is any closed contour which does not pass through the point z_0. State the corresponding result used. (4+2.5=6.5)

(b) Use Cauchy Gourset theorem to evaluate :

(i) $\int_C f(z)dz$, when $f(z) = \frac{1}{z^2+2z+2}$ and C is the unit circle $|z| = 1$ in either direction.

(ii) $\int_c f(z)dz$, when $f(z)=\frac{5z+7}{z^2+2z-3}$ and C is the circle $|z-2| = 2$. (3+3.5=6.5)

(c) State and prove Cauchy Integral Formula. (2+4.5=6.5)

(d) Evaluate the following integrals :

(i) $\int_c \frac{\cos z}{z(z^2+8)}dz$, where C is the positive oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

(ii) $\int_c \frac{2s^2-s-2}{s-2}ds$, $|z| \neq 3$ at $z = 2$, where C is the circle $|z| = 3$. (3.5+3=6.5)

5. (a) If a series of complex numbers converges then prove that the nth term converges to zero as n tends to infinity. Is the converse true? Justify. (6.5)

(b) Find the Maclaurin series for the function $f(z) = \sinh z$. (6.5)

(c) If a series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to f(z) at all points interior to some circle $|z - z_0| = R$, then prove that it is the Taylor series for the function f(z) in powers of $z - z_0$. (6.5)

(d) Find the integral of f(z) around the positively oriented circle $|z| = 3$ when $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$. (6.5)

6. (a) For the given function $f(z) = \left(\frac{z}{2z+1}\right)^3$, show any singular point is a pole. Determine the order of each pole and find the corresponding residue. (6)

(b) Find the Laurent Series that represents the function $f(z) = z^2 \sin\frac{1}{z^2}$ in the domain $0 < |z| < \infty$. (6)

(c) Suppose that $z_n = x_n + iy_n$, $(n = 1,2,3,...)$ and $S = X + iY$. Then show that

$$\sum_{n=1}^{\infty} z_n = S \text{ iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

(6)

(d) If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then show that

$$\int_C f(z)\,dz = 2\pi i\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$ (6)